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The Uniqueness of Norm-One Projection in James-Type Spaces

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For the James-type space $\mathscr{X}_{\mathscr{F}}$ generated by a sequence of functions $\mathscr{F} = \{f_n\}_{n \in \mathbb{N}}$ we present a sufficient and necessary condition under which there exists a unique minimal projection from $\mathscr{X}_{\mathscr{F}}$ onto $\mathscr{Y}_{\mathscr{F}} = \mathscr{X}_{\mathscr{F}} \cap c_0$.

0. INTRODUCTION

Let $\mathscr{F} = \{f_n\}_{n \in \mathbb{N}}$, be a sequence of convex functions $f_n: \mathbb{R}^+ \to \mathbb{R}^+$ such that $f_n(0) = 0$ and $f_n/_{(0, +\infty)} > 0$, for every $n \in \mathbb{N}$. A sequence of functions with the above properties will be called an *Orlicz sequence*.

Let $\mathscr{F} = \{f_n\}$ be an Orlicz sequence. For any sequence of real numbers $\mathbf{x} = \{x_n\}$ put

$$\rho_{\mathscr{F}}(\mathbf{x}) = \sum_{n=1}^{\infty} f_n(|x_n|).$$

Then a Musielak–Orlicz sequence space is defined by

$$\ell_{\mathscr{F}} = \{\mathbf{x} = \{x_n\}_{n \in \mathbb{N}} : \lim_{\lambda \to 0} \rho_{\mathscr{F}}(\lambda \mathbf{x}) = 0\}.$$

We can equip $\ell_{\mathscr{F}}$ with the Luxemburg norm

$$\|\mathbf{x}\|_{\mathscr{F}} = \inf\{d > 0 : \rho_{\mathscr{F}}(\mathbf{x}/d) \leqslant 1\}.$$

For basic facts concerning Musielak–Orlicz spaces the reader is reffered to [10].

Now fix any sequence of real numbers $\mathbf{x} = \{x_n\}$, $m \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$, $1 \leqslant j_1 < \cdots < j_{2m+1}$, and put

$$\mathbf{x}_{j_1, \ldots, j_{2m+1}} = (x_{j_2} - x_{j_1}, \ldots, x_{j_{2m}} - x_{j_{2m-1}}, x_{j_{2m+1}}, 0, \ldots).$$

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DEFINITION 0.1. Let $\mathcal{X}_{\mathcal{F}} = \{\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} \in c : \|\mathbf{x}\| < +\infty\}$ where

$$\|\mathbf{x}\| = \sup\{\|\mathbf{x}_{j_1, \ldots, j_{2m+1}}\|_{\mathcal{F}} : m \in \mathbb{N}^*, 1 \leqslant j_1 < \cdots < j_{2m+1}\}.$$

Then the space $(\mathcal{X}_{\mathcal{F}}, \|\cdot\|)$ will be called the James space generated by $\mathcal{F}$.

Put $\mathcal{Y}_{\mathcal{F}} = \mathcal{X}_{\mathcal{F}} \cap c_0$. Note that if for all $n \in \mathbb{N}$ $f_n(t) = t^2$ then $\mathcal{Y}_{\mathcal{F}}$ is exactly the famous James space introducted in [5] and $\mathcal{X}_{\mathcal{F}} = \mathcal{Y}_{\mathcal{F}}^{**}$. For other generalizations of the James space see, e.g., [17].

Let $\mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$ denote the set of all linear projections from $\mathcal{X}_{\mathcal{F}}$ onto $\mathcal{Y}_{\mathcal{F}}$, i.e.,

$$\mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}}) = \{P \in \mathcal{L}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}}) : P/_{\mathcal{Y}_{\mathcal{F}}} = \mathrm{id}_{\mathcal{Y}_{\mathcal{F}}}\}.$$

A projection $P_0 \in \mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$ is called *minimal* if

$$\|P_0\| = \lambda(\mathcal{Y}_{\mathcal{F}}, \mathcal{X}_{\mathcal{F}}) = \inf\{\|P\| : P \in \mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})\}.$$

The constant $\lambda(\mathcal{Y}_{\mathcal{F}}, \mathcal{X}_{\mathcal{F}})$ is called the *relative projection constant.*

Note that the problem of finding a minimal projection, from a Banach space X onto a subspace Y, is strictly related to the Hahn–Banach extension theorem, because we look for an extension of the id: $Y \to Y$ to X of minimal norm.

For more information concerning minimal projection (existence, effective formulas, uniqueness or estimates of the norm) the reader is referred to [2–4, 6, 8, 11, 12, 14, 16].

Now, take $P_0 \in \mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$ given by

$$P_0\mathbf{x} = \mathbf{x} - (\lim_{n\to\infty} x_n) \cdot (1, 1, \ldots).$$

The aim of this paper is to characterize those James spaces $\mathcal{X}_{\mathcal{F}}$, for which P_0 is the unique minimal projection onto $\mathcal{Y}_{\mathcal{F}}$ (see Theorem 2.5). We also prove that for any Orlicz sequence $\mathcal{F}$ $\|P_0\| = 1$ (see Theorem 2.2). This generalizes the results from [13, 9] concerning the case when $\mathcal{F}$ is a constant sequence.

Now we present some results and definitions which will be of use later. Let $\mathcal{F}$ be an Orlicz sequence and let

$$\rho(\mathbf{x}) = \sup\{\rho_{\mathcal{F}}(\mathbf{x}_{j_1, \ldots, j_{2m+1}}) : m \in \mathbb{N}^*, 1 \leqslant j_1 < \cdots < j_{2m+1}\}. \tag{0.1}$$

We will refer to it as $\mathcal{F}$-modular.

Remark 0.2. Let $\mathcal{F}$ be an Orlicz sequence. Then for arbitrary $\mathbf{x} = \{x_n\}$:

(1) for every $m \in \mathbb{N}^*$, $1 \leqslant j_1 < \cdots < j_{2m+1}$ $\mathbf{x}_{j_1, \ldots, j_{2m+1}} \in \ell_{\mathcal{F}}$;

(2) $\|\mathbf{x}_{j_1, \ldots, j_{2m+1}}\| = \min\{M > 0 : \rho_{\mathcal{F}}(\mathbf{x}_{j_1, \ldots, j_{2m+1}}/M) \leqslant 1\}$.

Applying Remark 0.2 we obtain that $\mathcal{X}_{\mathcal{F}}$ is a Banach space, and $\|\cdot\|$ is a norm. Moreover, we have

LEMMA 0.3. *For arbitrary* $\mathbf{x} \in \mathcal{X}_{\mathcal{F}}$, $\rho(\mathbf{x}) \leqslant 1$ *if and only if* $\|\mathbf{x}\| \leqslant 1$.

Now we present some properties of convex functions, which can be found in [7, Chap. VII] (see also [1, 15]).

In the following theorems J denotes an open (not necessarily bounded) interval.

THEOREM 0.4. *For function* $f\colon J \to \mathbb{R}$ *the following conditions are equivalent*:

(1) *function* f *is convex*;

(2) *for any* $x_1 < x_2 < x_3$, $(x_3 - x_1) f(x_2) \leqslant (x_2 - x_1) f(x_3) + (x_3 - x_2) f(x_1)$;

(3) *for any* $x_1 < x_2 < x_3$, $(f(x_2) - f(x_1))/(x_2 - x_1) \leqslant (f(x_3) - f(x_1))/(x_3 - x_1)$;

(4) *for any* $x_1 < x_2 < x_3$, $(f(x_3) - f(x_1))/(x_3 - x_1) \leqslant (f(x_3) - f(x_2))/(x_3 - x_2)$.

THEOREM 0.5. *Let* $f\colon J \to \mathbb{R}$ *be a convex function. Then the coresponding function* I *defined by* $I(x, h) = (f(x+h) - f(x))/h$ *is increasing with respect to each variable.*

COROLLARY 0.6. *Let* $f\colon J \to \mathbb{R}$ *be a convex function. Then for arbitrary* $u \geqslant v \geqslant 0$ *function* $g(x) = f(x+u) - f(x+v)$ *is increasing.*

THEOREM 0.7. *Let* $f\colon J \to \mathbb{R}$ *be a convex function. Then for every* $x \in J$ *there exists the right derivative* $f'_{+}(x)$, *and the left derivative* $f'_{-}(x)$. *Moreover for all* $x, y \in J$ $x < y$:

(1) $f'_{-}(x) \leqslant f'_{-}(y)$, $f'_{+}(x) \leqslant f'_{+}(y)$ *and* $f'_{-}(x) \leqslant f'_{+}(x)$;

(2) $\lim_{t \to x^+} f'_{+}(t) = \lim_{t \to x^+} f'_{-}(t) = f'_{+}(x)$ *and* $\lim_{t \to x^-} f'_{+}(t) = \lim_{t \to x^-} f'_{-}(t) = f'_{-}(x)$.

THEOREM 0.8. *Let* $f_n : J \to \mathbb{R}$ *be a sequence of convex functions, let* Δ *be a dense subset of* J. *Suppose that for every* $n \in \mathbb{N}$:

(1) $\sup_n f_n(x) < +\infty$, *for every* $x \in \Delta$.

(2) $\inf_n f_n(x) > -\infty$, *for an* $x_0 \in J$.

Then for every compact set $E \subset J$ there is $M > 0$ such that each f_n, restricted to E, satysfies a Lipschitz condition with M.

THEOREM 0.9. *Let $f_n : J \to \mathbb{R}$ be a sequence of convex functions. If the sequence $\{f_n\}$ converges pointwise on J to a finite function f, then f is convex.*

THEOREM 0.10. *Let $f_n : J \to \mathbb{R}$ be a sequence of convex functions, and let Δ be a dense subset of J. If the sequence $\{f_n(x)\}$ converges (to a finite limit) for every $x \in \Delta$, then the sequence $\{f_n\}$ converges uniformly on every compact subset of J.*

COROLLARY 0.11. *Let $f_n : J \to \mathbb{R}$ be a sequence of convex functions. If the sequence $\{f_n\}$ converges in J to a finite function f, then f is convex. Moreover the sequence $\{f_n\}$ converges uniformly to f on every compact subset of J.*

THEOREM 0.12. *Let $f_n : J \to \mathbb{R}$ be a sequence of convex functions. If the sequence $\{f_n\}$ converges pointwise on J to a finite function f, then for arbitrary sequence $\{x_n\} \subset J$, $x_n \to x_0 \in J$*

$$\limsup_{n \to \infty} (f_n)'_+(x_n) \leqslant f'_+(x_0).$$

1. TECHNICAL RESULTS

Let $\mathscr{F} = \{f_n\}$ be an Orlicz sequence. To the end of this section, putting $f_n(0) = 0$ for $x < 0$, we can treat each f_n as a function defined on $\mathbb{R}$.

Now let us define the following auxiliary functions:

$$\psi = \sup_{n \in \mathbb{N}} f_n, \qquad \psi_n = \sup_{1 \leqslant i \leqslant n} f_i, \qquad \varphi = \limsup_{n \to \infty} f_n, \qquad \varphi_n = \sup_{i \geqslant n} f_i. \tag{1.1}$$

DEFINITION 1.1. We will call an Orlicz sequence $\mathscr{F} = \{f_n\}$ a proper Orlicz sequence, if the function ψ is locally bounded at zero. Otherwise, this sequence will be called degenerate.

LEMMA 1.2. *Let $\mathscr{F} = \{f_n\}$ be a degenerate Orlicz sequence. Then for every $x > 0$ $\psi(x) = +\infty$.*

Proof. Since functions f_n are increasing, ψ is also increasing. Note that ψ is not locally bounded at zero. There is a sequence $\{x_n\} \to 0^+$ such that $\lim_{n \to \infty} \psi(x_n) = +\infty$. Take any $x > 0$. Since $\psi(x_n) \leqslant \psi(x)$ for n sufficiently large, the lemma is proved. ∎

THEOREM 1.3. *Let $\mathscr{F}=\{f_n\}$ be a proper Orlicz sequence. Then there is an interval $I=(-\infty, d_0)$, where $d_0>0$, such that*:

(1) *ψ and φ are finite and convex on I;*

(2) *ψ_n converges uniformly to ψ on every compact contained in I;*

(3) *for every $n\in\mathbb{N}$, φ_n is convex;*

(4) *φ_n converges uniformly to φ on every compact contained in I.*

Proof. Since ψ is bounded on $I=(-\infty, d_0)$ for some $d_0>0$, ψ is a finite function on I. Let us define $\phi_{k,n}=\sup_{k\leqslant i\leqslant k+n} f_i$. It is clear that $\psi_n\to\psi$ and $\phi_{k,n}\to\varphi_k$ pointwise on I. Since for each k $\varphi_k\leqslant\psi$, φ_k is a finite function on I. Moreover ψ_n, $\phi_{k,n}$ are convex. Thus by Corollary 0.11, ψ, φ_k are convex on I, and also ψ_n converges uniformly to ψ on every compact contained in I. Since $\varphi\leqslant\psi$, φ is a finite function on I. We also know that $\varphi_k\to\varphi$ pointwise on I, so from the previous considerations it follows that φ_k are convex on I. Thus by Corollary 0.11 function φ is also convex on I, and in addition φ_n converges uniformly to φ on every compact contained in I. ∎

From Theorem 0.8 we immediately get

COROLLARY 1.4. *If $\mathscr{F}=\{f_n\}$ is a proper Orlicz sequence, then there exists $\varphi'_+(0)$.*

LEMMA 1.5. *Let $I=(-\infty, d_0)$, where $d_0>0$. Let $\mathscr{G}$ be a sequence of convex functions $g_n: I\to\mathbb{R}^+$ such that $g_n(0)=0$, $g_n/_{(0,d_0)}>0$. Assume furthermore that $\{g_n\}$ converges pointwise on I to a convex function g. Then for any sequence $\{x_m\}\subset(0, d_0)$, $x_m\to 0$ and for any $\varepsilon>0$ there exists n_0 such that inequality*

$$(g_n)'_+(x_m)-g'_+(x_m)<\varepsilon$$

holds for any $m\in\mathbb{N}$ and $n\geqslant n_0$.

Proof. Suppose, to the contrary, that for some sequence $x_m\to 0^+$ and for some $\varepsilon>0$ inequality

$$(g_{n_k})'_+(x_k)-g'_+(x_k)\geqslant\varepsilon \tag{1.2}$$

holds for a certain subsequence $n_k\to+\infty$ and $k\in K\subset\mathbb{N}$.

There are two possibilities:

(1^0) $\{x_k: k\in K\}$ is an infinite set.

Passing to the subsequence, if neccessary, we may assume that $x_k \to 0$. By Theorem 0.7

$$\lim_{k \to \infty} g'_+(x_k) = g'_+(0) \tag{1.3}$$

and by Theorem 0.12

$$\limsup_{k \to \infty} (g_{n_k})'_+(x_k) \leqslant g'_+(0). \tag{1.4}$$

Hence by (1. 2) we also have $g'_+(0) \geqslant \varepsilon + g'_+(0)$, a contradiction.

(2^0) $\{x_k: k \in K\}$ is a finite set.

Without loss, we can assume that $(g_{n_k})'_+(x) \geqslant g'_+(x) + \varepsilon$, for some $x \in [0, d_0)$ which, by Theorem 0.12, leads to a contradiction. ∎

COROLLARY 1.6. *Let $\mathscr{F} = \{f_n\}$ be a proper Orlicz sequence, and take $I = (-\infty, d_0)$ from Theorem* 1.3. *Then for any sequence $\{x_m\} \subset (0, d_0)$, $x_m \to 0$ and for any $\varepsilon > 0$ there exists n_0 such that the inequality*

$$(\varphi_n)'_+(x_m) - \varphi'_+(x_m) < \varepsilon$$

holds for any $m \in \mathbb{N}$ and $n \geqslant n_0$. Here φ_n and φ are functions defined by (1.1).

Proof. A sequence $\{\varphi_n\}$ converges pointwise on $\mathbb{R}$ to a function φ, which by Theorem 1.3 is finite and convex on I. Thus a sequence $\mathscr{G} = \{\varphi_n/_I\}$ fulfills the assumptions of Lemma 1.5. ∎

LEMMA 1.7. *Let $\mathscr{F} = \{f_n\}$ be a proper Orlicz sequence. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that for arbitrary sequence $\{d_m\} \subset (0, \delta)$, $d_m \to 0$,*

$$\varphi_n(d_m) - \varphi(d_m) < \varepsilon\, d_m$$

holds for any $m \in \mathbb{N}$ and $n \geqslant n_0$ (here n_0 depends on $\{d_m\}$).

Proof. Fix $\varepsilon > 0$. Take $I = (-\infty, d_0)$. By Theorem 1.3, φ is finite and convex on I. By Theorem 0.7 there is $\delta \in (0, d_0)$ such that for any $x \leqslant \delta$

$$\varphi'_+(x) - \varphi'_+(0) < \varepsilon/2. \tag{1.5}$$

Take any sequence $d_m \to 0^+$ contained in $(0, \delta)$. By Corollary 1.6, we get

$$(\varphi_n)'_+(d_m) - \varphi'_+(d_m) < \varepsilon/2 \tag{1.6}$$

for any $n \geqslant n_0$ and $m \in \mathbb{N}$.

Combining (1.5) and (1.6) we obtain

$$(\varphi_n)'_+(d_m)-\varphi'_+(0)=[(\varphi_n)'_+(d_m)-\varphi'_+(d_m)]-[\varphi'_+(d_m)-\varphi'_+(0)]$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon. \tag{1.7}$$

Hence

$$(\varphi_n)'_+(d_m)<\varepsilon+\varphi'_+(0) \tag{1.8}$$

for any $n \geqslant n_0$ and $m \in \mathbb{N}$.

By Theorem 1.3 functions φ, φ_n are finite and convex on I. Applying Theorem 0.4 and Theorem 0.7 we get

$$\varphi'_+(0) \leqslant \frac{\varphi(d_m)}{d_m}, \qquad \text{for any} \quad n \in \mathbb{N}$$

and

$$(\varphi_n)'_+(d_m) \geqslant (\varphi_n)'_-(d_m) \geqslant \frac{\varphi_n(d_m)}{d_m}, \qquad \text{for any} \quad m, n \in \mathbb{N}.$$

Consequently, by (1.8)

$$\frac{\varphi_n(d_m)}{d_m} \leqslant (\varphi_n)'_+(d_m)<\varepsilon+\varphi'_+(0)<\varepsilon+\frac{\varphi(d_m)}{d_m}$$

for any $n \geqslant n_0$ and $m \in \mathbb{N}$, which gives the result. ∎

THEOREM 1.8. *Let $\mathscr{F}=\{f_n\}$ be a proper Orlicz sequence, such that $\varphi'_+(0)=0$. Take convex functions $h_1, ..., h_s : \mathbb{R}^+ \to \mathbb{R}^+$ with $h_i(0)=0$ and $h_i/_{(0,+\infty)}>0$. Then for any $c>0$, $b>0$ there is $\delta>0$ such that for arbitrary $\{d_m\} \subset (0, \delta)$, $d_m \to 0$ there exists n_0 such that*

$$h_i(b+c\,d_m)>h_i(b)+\varphi_n(d_m)$$

for any $i \in \{1, ..., s\}$, $n \geqslant n_0$ and $m \in \mathbb{N}$.

Proof. By Theorem 0.4 we get

$$h_i(b+cx)-h_i(b) \geqslant (h_i)'_+(b)\cdot cx, \qquad \text{for any} \quad i \in \{1, ..., s\}, \quad \text{and} \quad x>0. \tag{1.9}$$

Put $\varepsilon := \min_{i \in \{1, ..., s\}} \{(h_i)'_+(b) \cdot c\}$. Note that $c > 0$ and $(h_i)'_+(b) > 0$, for each i. Hence $\varepsilon > 0$. Since $\lim_{x \to 0^+} (\varphi(x)/x) = \varphi'_+(0) = 0$, there is $\delta_1 > 0$ such that

$$\varphi(x) < \frac{\varepsilon}{2} x, \qquad \text{for any} \quad x < \delta_1. \tag{1.10}$$

For $\varepsilon/2$ choose δ_2 from Lemma 1.7. Put $\delta = \min\{\delta_1, \delta_2\}$.

Now take any sequence $d_m \to 0^+$ contained in $(0, \delta)$. By Lemma 1.7 and (1.10),

$$\varphi_n(d_m) = [\varphi_n(d_m) - \varphi(d_m)] + \varphi(d_m) < \frac{\varepsilon}{2} d_m + \frac{\varepsilon}{2} d_m = \varepsilon\, d_m \tag{1.11}$$

for any $n \geqslant n_0$ and $m \in \mathbb{N}$.

By (1.9) and (1.11),

$$h_i(b + c\, d_m) - h_i(b) \geqslant \varepsilon\, d_m > \varphi_n(d_m)$$

for any $i \in \{1, ..., s\}$, $n \geqslant n_0$ and $m \in \mathbb{N}$. ∎

THEOREM 1.9. *Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function with properties: $h(0) = 0$ and $h/_{(0, +\infty)} > 0$. Let $g\colon \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ be a function for which there is $d_0 > 0$ (d_0 can be ∞) such that a function g is finite, convex and $g/_{(d_0, +\infty)} = +\infty$. Assume furthermore that $g(0) = 0$, $g/_{(0, d_0)} > 0$, $g'_+(0) > 0$ and g is increasing on $\mathbb{R}$. Then there is $c \in (0, 1)$ such that for any $b \in \mathbb{R}$, $d \in \mathbb{R} \backslash \{0\}$ with $h(|b|) < 2$, $h(|d|) < 2$,*

$$h(|b + c\, d|) < h(|b|) + g(|d|).$$

Proof. Suppose $b \geqslant 0$, $d > 0$. Assume $\beta = g'_+(0) > 0$. By Theorem 0.4 and by $\lim_{x \to d_0^+} g(x) \leqslant g(d_0)$,

$$\beta = g'_+(0) \leqslant \frac{g(x)}{x}, \qquad \text{for every} \quad x \in \mathbb{R}^+. \tag{1.12}$$

Note that $0 < d < x_0$, $0 < b < x_0$ where x_0 is such that $h(x_0) > 2$. Hence by Theorem 0.8, h fulfills a Lipschitz condition on $[-2x_0, 2x_0]$ with a constant M. Take $c \in (0, 1)$ such that $c < \beta/M$. Then

$$h(b + cd) - h(b) \leqslant M \cdot cd < \beta d \leqslant g(d)$$

for any $b \in [0, x_0)$, $d \in (0, x_0)$.

Note that for any b, d

$$h(|b+c\,d|)<h(|b|+c\,|d|),$$

which completes the proof. ▮

Corollary 1.10. *Let $\mathcal{F}=\{f_n\}$ be a proper Orlicz sequence, with $\varphi'_+(0)>0$. Take a convex function $h\colon \mathbb{R}^+\to\mathbb{R}^+$ with properties: $h(0)=0$ and $h/_{(0,d_0)}>0$. Then there is $c\in(0,1)$ such that for any $b\in\mathbb{R}$, $d\in\mathbb{R}\backslash\{0\}$, $h(|b|)<2$, $h(|d|)<2$ we have*

$$h(|b+c\,d|)<h(|b|)+\varphi(|d|),$$

where φ is given by (1.1).

Proof. By Theorem 1.3 we can take the greatest number d_0 (or ∞) such that a function $\varphi/_{(-\infty,d_0)}$ is finite and convex. Then φ with d_0 satisfies the assumptions of Theorem 1.9. ▮

Lemma 1.11. *Take $P\in\mathcal{P}(\mathcal{X}_{\mathcal{F}},\mathcal{Y}_{\mathcal{F}})$ given by $P\mathbf{x}=\mathbf{x}-(\lim_{n\to\infty}x_n)\cdot\mathbf{y}$, for every $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$, where $\mathbf{y}=\{y_n\}_{n\in\mathbb{N}}\in\mathcal{X}_{\mathcal{F}}$ and $\lim_{n\to\infty}y_n=1$. Fix $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$. Then for any $\varepsilon>0$ there are $1\leqslant j_1<j_2$, M_0 such that for any $M\geqslant M_0$ we can choose $K_0(M)$, $j_3,\ldots,j_{2M}$ for which*

$$\rho_{\mathcal{F}}((P\mathbf{x})_{j_1,\ldots,j_{2M},k})>\rho(P\mathbf{x})-\varepsilon$$

holds for every $k\geqslant K_0(M)$.

Proof. The proof is tedious and uses only the continuity of the functions f_i, thus we omit it. ▮

Remark 1.12. $P\in\mathcal{P}(\mathcal{X}_{\mathcal{F}},\mathcal{Y}_{\mathcal{F}})$ is a norm-one projection if and only if for arbitrary $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$ $\rho(\mathbf{x})\leqslant 1$ implies $\rho(P\mathbf{x})\leqslant 1$.

Lemma 1.13. *Let $\mathcal{F}=\{f_n\}$ be a proper Orlicz sequence. Consider the sequence $\mathbf{x}=\{x_n\}$, such that $x_n=x$ for every $n\geqslant n_0$. Then $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$.*

Proof. The proof is routine, so we omit it. ▮

Remark 1.14. Let $\mathcal{F}=\{f_n\}$ be a proper Orlicz sequence. Then the following conditions are equivalent:

(1) $P\in\mathcal{P}(\mathcal{X}_{\mathcal{F}},\mathcal{Y}_{\mathcal{F}})$;

(2) P is of the form $P\mathbf{x}=\mathbf{x}-(\lim_{n\to\infty}x_n)\cdot\mathbf{y}$, for every $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$, where $\mathbf{y}=\{y_n\}_{n\in\mathbb{N}}\in\mathcal{X}_{\mathcal{F}}$ and $\lim_{n\to\infty}y_n=1$.

Proof. For a projection $P\in\mathcal{P}(\mathcal{X}_{\mathcal{F}},\mathcal{Y}_{\mathcal{F}})$ putting $\mathbf{y}=\mathbf{e}-P(\mathbf{e})$, where $\mathbf{e}=(1,1,1,\ldots)$, we get the result. ▮

2. MAIN RESULTS

If $\mathcal{F}=\{f_n\}$ is a degenerate Orlicz sequence, then by Lemma 1.5 we can easily get the following

Remark 2.1. Let $\mathcal{F}=\{f_n\}$ be a degenerate Orlicz sequence. Then $\mathcal{X}_{\mathcal{F}}=\mathcal{Y}_{\mathcal{F}}$ and consequently $\mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})=\{\mathrm{id}\}$. Therefore, we will further deal only with the case when $\mathcal{F}=\{f_n\}$ is a proper Orlicz sequence.

THEOREM 2.2. *Let $\mathcal{F}=\{f_n\}$ be a proper Orlicz sequence. Take $P_0\in\mathcal{P}(\mathcal{X}_{\mathcal{Y}}, \mathcal{Y}_{\mathcal{F}})$ given by $P_0\mathbf{x}=\mathbf{x}-(\lim_{n\to\infty} x_n)\cdot(1, 1, ...)$, for any $\mathbf{x}=\{x_n\}\in\mathcal{X}_{\mathcal{F}}$. Then $\|P_0\|=1$ and consequently P_0 is a minimal projection.*

Proof. In view of Remark 1.12, it is sufficient to show that for any $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$ inequality $\rho(\mathbf{x})\leqslant 1$ implies $\rho(P_0\mathbf{x})\leqslant 1$.

To do this take any $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}$ such that $\rho(\mathbf{x})\leqslant 1$ and $\lim_{n\to\infty} x_n=d\neq 0$. Fix any $\varepsilon>0$. By Lemma 1.11 there exist $M_0, j_1<\cdots<j_{2M}, K_0(M_0)$ such that

$$\rho(P_0\mathbf{x})-\varepsilon<\rho_{\mathcal{F}}((P_0\mathbf{x})_{j_1, ..., j_{2M_0}, k}), \qquad \text{for every} \quad k\geqslant K_0(M_0). \tag{2.1}$$

Since $|x_n|\to|d|$ and $|x_n-d|\to 0$, there is $k_1>K_0$ such that $|x_{k_1}|>|d|/2$ and $|x_{k_1}-d|<|d|/2$. Then

$$f_{M_0+1}(|x_{k_1}|)\geqslant f_{M_0+1}(|d|/2)\geqslant f_{M_0+1}(|x_{k_1}-d|). \tag{2.2}$$

Since $P_0\mathbf{x}=\{x_n-d\}_{n\in\mathbb{N}}$, it follows from (2.2) that

$$\rho_{\mathcal{F}}((\mathbf{x})_{j_1, ..., j_{2M_0}, k_1})\geqslant\rho_{\mathcal{F}}((P_0\mathbf{x})_{j_1, ..., j_{2M_0}, k_1}).$$

Consequently, by (2.1) we get

$$\rho_{\mathcal{F}}((\mathbf{x})_{j_1, ..., j_{2M_0}, k_1})\geqslant\rho_{\mathcal{F}}((P_0\mathbf{x})_{j_1, ..., j_{2M_0}, k_1})>\rho(P_0\mathbf{x})-\varepsilon.$$

Hence $\rho(P_0\mathbf{x})\leqslant\rho(\mathbf{x})$, which completes the proof. ∎

Now let us proceed to the proof of the main result. For this purpose let us make a usefull definition.

DEFINITION 2.3. Let $n_1, n_2, ..., n_{n_0+1}$ be fixed integers, all even or all odd, such that $n_{k+1}>n_k+6$. For arbitrary numbers b, d, e let us denote by $\mathbf{x}(b, d, e)$ the following sequence

$$\mathbf{x}(b, d, e)=(0, ..., 0, \overset{n_1}{\gamma b}, 0, ..., 0, \overset{n_2}{\gamma e}, 0, ..., 0, \overset{n_{n_0}}{\gamma e}, 0, ..., 0, \overset{n_{n_0+1}}{d}, d, ...), \tag{2.3}$$

where $\gamma\in\{-1, 1\}$. By Lemma 1.13, $\mathbf{x}(b, d, e)\in\mathcal{X}_{\mathcal{F}}$.

Now we will prove a crucial lemma.

LEMMA 2.4. *Let $\mathscr{F}=\{f_n\}$ be a proper Orlicz sequence. Take $\{n_k\}$ the same as in Definition 2.3. Then for any $0<b_1<b_2$ there exist $e>0$ and $d_1>0$ such that for all $b\in[b_1,b_2]$ and $d\leqslant d_1$ we can choose $1\leqslant l_1<\cdots<l_{n_0}$ for which*

(1) *if $\gamma=1$ then $\rho(\mathbf{x}(b,d,e))\leqslant f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{n_0}}(e)+2\varphi_{n_0}(d)$; moreover, for some $j_1,\ldots,j_{2m}$ the sequence $\mathbf{x}(b,d,e)_{j_1,\ldots,j_{2m}}$ has a form*

$$(0,\ldots,0,\underbrace{\overset{n_1}{0^*-\gamma b}}_{l_1\text{th coordinate}},0,\ldots,0,\underbrace{\overset{n_2}{0^*-\gamma e}}_{l_2\text{th coordinate}},0,\ldots,0,\underbrace{\overset{n_{n_0}}{0^*-\gamma e}}_{l_{n_0}\text{th coordinate}},0,0,\ldots);$$

(2) *if $\gamma=-1$ then $\rho(\mathbf{x}(b,d,e))\leqslant f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{n_0}-1}(e)+f_{l_{n_0}}(e+d)+2\varphi_{n_0}(d)$; moreover, for some $j_1,\ldots,j_{2m}$ the sequence $\mathbf{x}(b,d,e)_{j_1,\ldots,j_{2m}}$ has a form*

$$(0,\ldots,0,\underbrace{\overset{n_1}{0^*-\gamma b}}_{l_1\text{th coordinate}},0,\ldots,0,\underbrace{\overset{n_2}{0^*-\gamma e}}_{l_2\text{th coordinate}},0,\ldots,0,\underbrace{\overset{n_{n_0}+1}{d}-\overset{n_{n_0}}{\gamma e}}_{l_{n_0}\text{th coordinate}},0,0,\ldots),$$

(*the sequence $\mathbf{x}(b,d,e)_{j_1,\ldots,j_{2m}}$ defined above differs from a similar sequence described in* (1) *only on the coordinate l_{n_0}*).

Here the symbol $\overset{k}{z}$ denotes that z is taken from the kth coordinate of the sequence $\mathbf{x}(b,d,e)$, and $0^*-\overset{n}{z}$ is a shortened notation for $\overset{n+1}{0}-\overset{n}{z}$.

Proof. Let us denote by Γ the set of all triples $(\gamma_1,\gamma_2,\gamma_1')$ such that $\gamma_1\in\{0,1\}$, $\gamma_1'\in\{0,1\}$, $\gamma_2=1$ when $\gamma_1=0$, and $\gamma_2\in\{0,1\}$ when $\gamma_1=1$.

For fixed $(\gamma_1,\gamma_2,\gamma_1')\in\Gamma$ a sequence $l_1,\ldots,l_{k_0}$ $(k_0\leqslant n_0)$ will be called $(\gamma_1,\gamma_2,\gamma_1')$ *possible* if there exist $j_1<\cdots<j_{2m+1}$ such that

$$\rho_{\mathscr{F}}(\mathbf{x}(b,d,e)_{j_1,\ldots,j_{2m+1}})=f_{l_1}(\gamma_1 b-\gamma_2 e)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e-\gamma\gamma_1' d)$$
$$+\text{some elements of form } f_k(d),$$
$$\text{where}\quad k>l_{k_0}.$$

Let $\chi_k^{\gamma_1'}(d)=\sup_{k<k_1<k_2}\{f_{k_1}(d)+f_{k_2}((1-\gamma_1')\,d)\}$. By the definition of $\rho(\mathbf{x}(b,d,e))$, it is easy to see that

$$\rho(\mathbf{x}(b,d,e))=\max\{\max\{f_{l_1}(\gamma_1 b-\gamma_2 e)+f_{l_2}(e)+\cdots$$
$$+f_{l_{k_0}}(e-\gamma\gamma_1' d)+\chi_{l_{k_0}}^{\gamma_1'}(d),\text{ where }(\gamma_1,\gamma_2,\gamma_1')\in\Gamma$$
$$\text{and } l_1,\ldots,l_{k_0}\text{ is a }(\gamma_1,\gamma_2,\gamma_1')\text{ possible sequence}\},$$

$$\max\left\{f_{l_1}(\gamma_1 b-\gamma\gamma_1' d)+\chi_{l_1}^{\gamma_1'}(d),\ \text{where } \gamma_1,\gamma_2,\gamma_1'\in\Gamma \right.$$
$$\left.\text{and } l_1\leqslant\left[\frac{n_1+1}{2}\right]\right\},\ \{\chi_0^1(d)\}\Big\}. \tag{2.4}$$

Since numbers $l_1, ..., l_{k_0}$ appearing in possible sequences can be estimated from above by $[(n_{n_0}+1)/2]$, we can write above max instead of sup.

Now, consider the functions $\check{f}_k=\inf_{n\leqslant[(k+1)/2]} f_n$, $\hat{f}_k=\sup_{n\leqslant[(k+1)/2]} f_n$, $\varphi_n=\sup_{i\geqslant n} f_i$. (Here the symbol $[\alpha]$ denotes the greatest integer less or equal to α.)

Note that $\check{f}_k$ and $\hat{f}_k$ are convex for each k, moreover $\check{f}_k(0)=\hat{f}_k(0)=0$. Hence $\check{f}_k$ and $\hat{f}_k$ are also increasing.

Choose d_0 from Theorem 1.3. Then $\psi/_{(-\infty, d_0)}$ is finite and convex.

Now take $e\in\mathbb{R}$ for which

$$0<e<d_0/2 \qquad \text{and} \qquad n_0\psi(e)<\hat{f}_{n_1}(b_1). \tag{2.5}$$

By Theorem 0.8, $\hat{f}_i$ fulfills a Lipschitz condition on $[-(b_2+1), b_2+1]$. Hence, by (2.5), there is d_1 such that for any $d\leqslant d_1, b\geqslant b_1$ and any $\gamma_1,\gamma_2,\gamma_1'$:

$$\begin{aligned}
&(1)\quad \hat{f}_{n_1}(\gamma_1 b-\gamma\gamma_1' d)+2\psi(d)<\hat{f}_{n_1}(\gamma_1 b)+\check{f}_{n_{n_0}}(e);\\
&(2)\quad \text{for any } i\leqslant\left[\frac{n_{n_0}+1}{2}\right]\ f_i(e+d)-f_i(e)+2\psi(d)<\check{f}_{n_{n_0}}(e);\\
&(3)\quad (n_0-1)\,\psi(e)+\psi(e+d)+2\psi(d)<\hat{f}_{n_1}(b_1);\\
&(4)\quad 2\psi(d)<\check{f}_{n_{n_0}}(e).
\end{aligned} \tag{2.6}$$

We divide our proof into two steps.

Step I. The following equality holds

$$\begin{aligned}
\rho(\mathbf{x}(b,d,e))=\max\{&f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e-\gamma\gamma_1' d)+\chi_{l_{k_0}}^{\gamma_1'}(d),\\
&\text{where } \gamma_1'\in\{0,1\} \text{ and } l_1, ..., l_{k_0}\\
&\text{is a } (1,0,\gamma_1') \text{ possible sequence}\}.
\end{aligned}$$

For this purpose let us make some estimates.

(1) For any γ_1,γ_1' and $l_1\leqslant[(n_{n_0}+1)/2]$

$$\begin{aligned}
\max\{f_{l_1}(\gamma_1 b-\gamma\gamma_1' d)+\chi_{l_1}^{\gamma_1'}(d), \chi_0^1(d)\}&\leqslant\hat{f}_{n_1}(b-\gamma\gamma_1' d)+2\psi(d)\\
&<\hat{f}_{n_1}(b)+\check{f}_{n_{n_0}}(e)\leqslant f_{l_1'}(b)+f_{l_1'+1}(e).
\end{aligned}$$

The last equality holds for l'_1 such that $\hat{f}_{n_1}(b) = f_{l'_1}(b)$ and $l'_1 \leqslant [(n_{n_0}+1)/2]$. Note that the sequence l'_1, l'_1+1 is $(1, 0, 0)$ possible.

(2) For any system $(0, 1, \gamma'_1)$ and $(0, 1, \gamma'_1)$ possible sequence $l_1, ..., l_{k_0}$ we have

$$\begin{aligned} & f_{l_1}(e) + \cdots + f_{l_{k_0}}(e - \gamma\gamma'_1 d) + 2\psi(d) \\ & \quad \leqslant (k_0 - 1)\, \psi(e) + \psi(e+d) + 2\psi(d) \\ & \quad \leqslant (n_0 - 1)\, \psi(e) + \psi(e+d) + 2\psi(d) < \hat{f}_{n_1}(b_1) \leqslant \hat{f}_{n_1}(b) = f_{l'_1}(b), \end{aligned}$$

where $l'_1 \leqslant [(n_1+1)/2]$. It is clear that the sequence l'_1 is $(1, 0, 0)$ possible.

(3) For any system $(1, 1, \gamma'_1)$ and $(1, 1, \gamma'_1)$ possible sequence $l_1, ..., l_{k_0}$ we have

$$\begin{aligned} & f_{l_1}(b-e) + \cdots + f_{l_{k_0}}(e - \gamma\gamma'_1 d) + f_{k_1}(d) + f_{k_2}((1-\gamma'_1)\, d) \\ & \quad < f_{l_1}(b) + \cdots + f_{l_{k_0}}(e - \gamma\gamma'_1 d) + f_{k_1}(d) + f_{k_2}((1-\gamma'_1)\, d) \end{aligned}$$

for any $k_2 > k_1 > l_{k_0}$.

Let $\rho_{\mathscr{F}}(\mathbf{x}(b, d, e)_{j_1, ..., j_{2m+1}})$ has the form of the left side above inequality. Assume furthermore that e appearing in the factor $f_{l_1}(b-e)$ is taken from the n_jth coordinate in the sequence $\mathbf{x}(b, d, e)$. Between the n_1th coordinate and the n_jth coordinate in the sequence $\mathbf{x}(b, d, e)$ there is at least one zero. Taking this zero and putting it in place of earlier mentioned e in the sequence $\mathbf{x}(b, d, e)_{j_1, ..., j_{2m+1}}$ we get a sequence which $\mathscr{F}$-modular (see (0.1)) is equal to the right side above inequality. Thus sequence $l_1, ..., l_{k_0}$ is $(1, 0, \gamma'_1)$ possible.

Step II. If $l_1 < \cdots < l_{k_0}$ is a $(1, 0, \gamma'_1)$ possible sequence, then there are $l'_2, ..., l'_{n_0}$, $l_1 < l'_2 < \cdots < l'_{n_0}$ and $\{l_1, l_2, ..., l_{k_0}\} \subset \{l_1, l'_2, ..., l'_{n_0}\}$. Moreover, the sequence $l_1, l'_2, ..., l'_{n_0}$ is, for any $\gamma''_1 \in \{0, 1\}$, $(1, 0, \gamma''_1)$ possible, also there are $j_1, ..., j_{2m}$ such that $\mathbf{x}(b, d, e)_{j_1, ..., j_{2m}}$ has a form

$$\begin{aligned} & (0, ..., 0, \underbrace{\overset{n_1}{0^* - \gamma b}}_{l_1\text{th coordinate}}, 0, ..., 0, \underbrace{\overset{n_2}{0^* - \gamma e}}_{l'_2\text{th coordinate}}, ..., \underbrace{\overset{n_{n_0-1}}{0^* - \gamma e}}_{l'_{n_0-1}\text{th coordinate}}, 0, ..., 0, \\ & \quad \underbrace{\overset{n_{n_0}}{\mathbf{x}(b, d, e)_k - \gamma e}}_{l'_{n_0}\text{th coordinate}}, 0, ...) \end{aligned}$$

for any $k > n_0$.

To do this, take a $(1, 0, \gamma'_1)$ possible sequence $l_1, ..., l_{k_0}$. Then there exist $j_1, ..., j_{2m+1}$ such that $\rho_{\mathscr{F}}(\mathbf{x}(b, d, e)_{j_1, ..., j_{2m+1}}) = f_{l_1}(b) + f_{l_2}(e) + \cdots + f_{l_{k_0}}(e - \gamma\gamma'_1 d)$. If in the sequence $\mathbf{x}(b, d, e)_{j_1, ..., j_{2m+1}}$, e (or b) which appears on the l_ith (resp. l_{i+1}th) coordinate is taken from the n_pth (resp. n_qth) coordinate of the sequence $\mathbf{x}(b, d, e)$, then $l_{i+1} - l_i \leqslant (n_q - n_p)/2 - 1$.

Consider a sequence $\mathbf{x}(b, d, e)_{1, 2, ..., 2[(n_1-1)/2], n_1, ..., n_{n_0}, k}$, where after n_1 there are all numbers in succession up to n_0. In this sequence between the terms of forms $0^* - \gamma \overset{n_p}{e}$ (or $0^* - \gamma \overset{n_1}{b}$) and $0^* - \gamma \overset{n_q}{e}$ there are exactly $(n_q - n_p)/2 - 1$ coordinates (having a form $0 - 0$ or $0 - \gamma e$). Thus by removing a proper number of systems having a form $0 - 0$ or $0 - \gamma e$ we get a sequence $\mathbf{x}(b, d, e)_{j'_1, ..., j'_{2m_1}}$, which has on the l_ith coordinate $(i > 1)$ term $0^* - \gamma e$, and on the l_1th term $0^* - \gamma b$.

Assume that this sequence (i.e., $\mathbf{x}(b, d, e)_{j'_1, ..., j'_{2m_1}}$) has t coordinates of forms $0^* - \gamma b$ or $0^* - \gamma e$, and designate them successively by $s_1, ..., s_t$ (obviously $l_1 = s_1$ and $\{l_2, ..., l_{k_0}\} \subset \{s_1, ..., s_t\}$).

Fix coordinates s_i and s_{i+1} $(i \in \{1, ..., t-1\})$, assume that a non-zero term (i.e., γb or γe) on the s_ith (resp. s_{i+1}th) coordinate in the sequence $\mathbf{x}(b, d, e)_{j'_1, ..., j'_{2m_1}}$ is taken from the n_pth (resp. n_qth) coordinate of the sequence $\mathbf{x}(b, d, e)$.

There exists $u \in \{p+1, ..., q\}$ such that

$$\frac{n_{u-1} - n_p}{2} < s_{i+1} - s_i \leqslant \frac{n_u - n_p}{2}. \tag{2.7}$$

If $j'_{2\alpha+1} = n_p$ and $j'_{2\beta+1} = n_q$, then considering the sequence

$$\mathbf{x}(b, d, e)_{j'_1, ..., j'_{2\alpha}, n_p, ..., n_{u+1}, j'_{2\beta+3}, ..., j'_{2m_1}}, \tag{2.8}$$

where after n_p appear successively all numbers up to n_{u+1}, we can see that in this sequence between coordinates in which there appear terms $0^* - \gamma \overset{n_p}{e}$ (or $0^* - \gamma \overset{n_1}{b}$) and $0^* - \gamma \overset{n_u}{e}$ there are $(n_u - n_p)/2 - 1$ coordinates, of which $u - p - 1$ have a form $0^* - \gamma e$. Since $(n_{u-1} - n_p)/2 \geqslant u - p - 1$ and (2.7) holds true, then by removing a proper numbers of systems of the form $0 - 0$ from the sequence (2.8) we will get the sequence $j''_1, ..., j''_{2m_1}$ such that $\mathbf{x}(b, d, e)_{j'_1, ..., j'_{2m_1}}$ is equal to $\mathbf{x}(b, d, e)_{j''_1, ..., j''_{2m_1}}$ on coordinates from 1 to s_i and from the coordinate s_{i+1} up. Moreover, in this sequence on coordinates from $s_i + 1$ to $s_{i+1} - 1$ there appear all terms of the form $0^* - \gamma \overset{n_1}{e}$, for all $j \in \{p+1, ..., u-1\}$, and on the coordinate s_{i+1} there is a term $0^* - \gamma \overset{n_u}{e}$.

Now applying this procedure to a sequence $\mathbf{x}(b, d, e)_{j'_1, ..., j'_{2m_1}}$ and coordinates s_1, s_2, we get a new sequence and applying to it the same procedure to coordinates s_2, s_3, we get the next sequence, and so on. Finally, we get a sequence $\mathbf{x}(b, d, e)_{j''_1, ..., j''_{2m_1}}$, which has t_1 coordinates of the form $0^* - \gamma e$ or $0^* - \gamma b$. Let these are the places $r_1 < r_2 < \cdots < r_{t_1}$ then $l_1 = s_1 = r_1$ and $\{l_2, ..., l_{k_0}\} \subset \{s_2, ..., s_t\} \subset \{r_2, ..., r_{t_1}\}$. Moreover on the coordinate r_1 there is a term $0^* - \overset{n_1}{\gamma b}$ and on the coordinate r_i (for $i=2, ..., t_1$) $0^* - \overset{n_j}{\gamma e}$. Now by completing a sequence $j''_1, ..., j''_{2m_1}$ to a sequence $j''_1, ..., j''_{2m_1}, n_{t_1+1}, n_{t_1+1}+1, ..., n_{n_0-1}, n_{n_0-1}+1, n_{n_0}, k$, where after the term j''_{2m_1} there appear successively all pair of terms from $n_{t_1+1}, n_{t_1+1}+1$ to $n_{n_0-1}, n_{n_0-1}+1$, we will get a sequence which has the properties required in Step II.

Now let us come back to the proof of lemma.

First we consider the case $\gamma=1$.

Let $l_1, l_2, ..., l_{k_0}$ $(k_0<n_0)$ be any $(1, 0, \gamma'_1)$ possible sequence. Choose for it a sequence $l_1, l'_2, ..., l'_{n_0}$ from Step II and assume that $l'_{i_0} \notin \{l_1, l_2, ..., l_{k_0}\}$. Then

$$\begin{aligned} & f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e-\gamma'_1 d)+\chi^{\gamma'_1}_{l_{k_0}}(d) \\ & \quad \leqslant f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e)+2\psi(d) \\ & \quad < f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e)+f_{l'_{i_0}}(e) \\ & \quad \leqslant f_{l_1}(b)+f_{l'_2}(e)+\cdots+f_{l'_{n_0}}(e). \end{aligned}$$

Thus by Step I and properties of $\{l_1, l'_2, ..., l'_{n_0}\}$ we get

$$\begin{aligned} \rho(\mathbf{x}(b, d, e)) &= \max\{f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e-\gamma'_1 d)+\chi^{\gamma'_1}_{l_{k_0}}(d), \\ &\qquad \text{where } \gamma'_1 \in \{0, 1\} \text{ and } l_1, ..., l_{k_0} \\ &\qquad \text{is a } (1, 0, \gamma'_1) \text{ possible sequence}\} \\ &= \max\{f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{n_0}}(e)+\chi^0_{l_{n_0}}(d), \\ &\qquad \text{where } l_1, ..., l_{n_0} \text{ is a } (1, 0, 0) \text{ possible sequence} \\ &\qquad \text{having properties required in Step II}\} \\ &= (\text{for a certain } (1, 0, 0) \text{ possible sequence } l^1_1, ..., l^1_{n_0} \\ &\qquad \text{having properties required in Step II}) \\ &= f_{l^1_1}(b)+f_{l^1_2}(e)+\cdots+f_{l^1_{n_0}}(e)+\chi^0_{l^1_{n_0}}(d) \\ &\leqslant f_{l^1_1}(b)+f_{l^1_2}(e)+\cdots+f_{l^1_{n_0}}(e)+2\psi_{n_0}(d). \end{aligned}$$

Hence the lemma is proved in this case.

Now, consider the second case, i.e., $\gamma=-1$.

Let $l_1, l_2, ..., l_{k_0}$ $(k_0<n_0)$ be any $(1, 0, \gamma'_1)$ possible sequence. Choose for it a sequence $l_1, l'_2, ..., l'_{n_0}$ from Step II and assume that $l'_{i_0} \notin \{l_1, l_2, ..., l_{k_0}\}$. Then

$$\begin{aligned} & f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e+\gamma'_1 d)+\chi^{\gamma'_1}_{l_{k_0}}(d) \\ & \quad \leqslant f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e+\gamma'_1 d)+2\psi(d) \\ & \quad < f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{k_0}}(e)+f_{l'_{i_0}}(e) \\ & \quad \leqslant f_{l_1}(b)+f_{l'_2}(e)+\cdots+f_{l'_{n_0}}(e+d). \end{aligned}$$

Thus by Step I and properties of $\{l_1, l'_2, ..., l'_{n_0}\}$ we get

$$\begin{aligned}\rho(\mathbf{x}(b, d, e)) &= \max\{f_{l_1}(b) + f_{l_2}(e) + \cdots + f_{l_{k_0}}(e + \gamma'_1 d) + \chi^{\gamma'_1}_{l_{k_0}}(d), \\ &\qquad \text{where } \gamma'_1 \in \{0, 1\} \text{ and } l_1, ..., l_{k_0} \\ &\qquad \text{is a } (1, 0, \gamma'_1) \text{ possible sequence}\} \\ &= \max\{f_{l_1}(b) + f_{l_2}(e) + \cdots + f_{l_{n_0}}(e + d) + \chi^0_{l_{n_0}}(d), \\ &\qquad \text{where } l_1, ..., l_{n_0} \text{ is a } (1, 0, 1) \text{ possible sequence} \\ &\qquad \text{having properties required in Step II}\} \\ &= (\text{for a certain } (1, 0, 1) \text{ possible sequence } l^1_1, ..., l^1_{n_0} \\ &\qquad \text{having properties required in Step II}) \\ &= f_{l^1_1}(b) + f_{l^1_2}(e) + \cdots + f_{l^1_{n_0}}(e + d) + \chi^0_{l^1_{n_0}}(d) \\ &\leqslant f_{l^1_1}(b) + f_{l^1_2}(e) + \cdots + f_{l^1_{n_0}}(e + d) + 2\psi_{n_0}(d).\end{aligned}$$

Hence the lemma is proved in this case, too. ∎

Now we are able to prove the following

THEOREM 2.5. *Let $\mathscr{F} = \{f_n\}$ be a proper Orlicz sequence. Take the "natural projection" $P_0 \in \mathscr{P}(\mathscr{X}_{\mathscr{F}}, \mathscr{Y}_{\mathscr{F}})$ defined in Theorem 2.2. Then P_0 is the unique minimal projection in $\mathscr{P}(\mathscr{X}_{\mathscr{F}}, \mathscr{Y}_{\mathscr{F}})$ if and only if $\varphi'_+(0) = 0$, where $\varphi = \limsup_{n \to \infty} f_n$.*

Proof. By Theorem 2.2, P_0 is a minimal projection in $\mathscr{P}(\mathscr{X}_{\mathscr{F}}, \mathscr{Y}_{\mathscr{F}})$

Suppose $\varphi'_+(0) = 0$ and take any $P \in \mathscr{P}(\mathscr{X}_{\mathscr{F}}, \mathscr{Y}_{\mathscr{F}})$. By Remark 1.14 there is $\mathbf{y} = \{y_n\} \in \mathscr{X}_{\mathscr{F}}$, $\lim_{n \to \infty} y_n = 1$ such that

$$P\mathbf{x} = \mathbf{x} - (\lim_{n \to \infty} x_n) \cdot \mathbf{y}, \qquad \text{for any} \quad \mathbf{x} = \{x_n\} \in \mathscr{X}_{\mathscr{F}}.$$

Now asume that $P \neq P_0$. There are two possible cases:

(1) There is a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ with properties:

For any $k \in \mathbb{N}$ $y_{n_k} \geqslant y_{n_k + 1}$ and $y_{n_k} \geqslant y_{n_{k+1}}$, moreover $y_{n_1} > y_{n_1 + 1}$.

Numbers $\{n_k\}$, $k \in \mathbb{N}$ are all even or all odd, and for any $k \in \mathbb{N}$ $n_{k+1} > n_k + 6$.

(2) There is a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ with properties:

For any $k \in \mathbb{N}$ $y_{n_k} \leqslant y_{n_k + 1}$ and $y_{n_k} \leqslant y_{n_{k+1}}$, moreover $y_{n_1} < y_{n_1 + 1}$.

Numbers $\{n_k\}$, $k \in \mathbb{N}$ are all even or all odd, and for any $k \in \mathbb{N}$ $n_{k+1} > n_k + 6$.

Put $c = |y_{n_1+1} - y_{n_1}| > 0$ and $\gamma = \operatorname{sgn}(y_{n_1+1} - y_{n_1})$.

Consider functions $\check{f}_k = \inf_{n \leqslant [(k+1)/2]} f_n$, $\hat{f}_k = \sup_{n \leqslant [(k+1)/2]} f_n$, $\varphi_n = \sup_{i \geqslant n} f_i$. (Here the symbol $[\alpha]$ denotes the greatest integer less or equal to α.)

Note that $\check{f}_k$ and $\hat{f}_k$ are convex for each k, moreover $\check{f}_k(0) = \hat{f}_k(0) = 0$. Also $\lim_{x \to \infty} \check{f}_k(x) = \lim_{x \to \infty} \hat{f}_k(x) = +\infty$. Therefore there are b_1, b_2, $0 < b_1 < b_2$ such that

$$\tfrac{1}{4} < \hat{f}_{n_1}(b_1) < \tfrac{1}{2} < 1 < \check{f}_{n_1}(b_2). \tag{2.9}$$

Choose d_0 from Theorem 1.3. Then $\psi/_{(-\infty, d_0)}$ is finite and convex.

For numbers $c > 0$, $b_1 > 0$ and functions $f_1, ..., f_{[(n_1+1)/2]}$ take $\delta > 0$ from Theorem 1.8. Take any sequence $d_\nu \to 0^+$, $d_\nu \leqslant \min\{\delta, d_0/2\}$. By Theorem 1.8 there is n_0 such that

$$f_i(b_1 + cd_\nu) > f_i(b_1) + 2\varphi_{n_0}(d_\nu)$$

for any $i \in \{1, ..., [(n_1+1)/2]\}$, and $\nu \in \mathbb{N}$.

By Corollary 0.6, a function $h_i(x) = f_i(x + cd) - f_i(x)$ is increasing, for fixed c, d, i, thus for $b \geqslant b_1$ we have $f_i(b + cd) - f_i(b) \geqslant f_i(b_1 + cd) - f_i(b_1) > 2\varphi_{n_0}(d)$. Hence we get

$$f_i(b + cd_\nu) > f_i(b) + 2\varphi_{n_0}(d_\nu) \tag{2.10}$$

for any $i \in \{1, ..., [(n_1+1)/2]\}$, $\nu \in \mathbb{N}$, and $b \in [b_1, b_2]$.

For $b_1 < b_2$ choose e, d_1 from Lemma 2.4. Since $d_\nu \to 0$ there is $d_{\nu_0} \leqslant d_1$.

For any $b \in [b_1, b_2]$ consider the sequence $\mathbf{x}(b) = \mathbf{x}(b, d_{\nu_0}, e)$ (see Definition 2.3). Let us remind that b, e, d_{ν_0} fulfill (2.5) and (2.6) (see the proof of Lemma 2.4).

By the formulas on P and $\mathbf{x}(b)$ we get

$$P\mathbf{x}(b) = (-d_{\nu_0} y_1, ..., -d_{\nu_0} y_{n_1-1}, \underbrace{\gamma b - y_{n_1} d_{\nu_0}}_{n_1}, \underbrace{-y_{n_1+1} d_{\nu_0}}_{n_1+1}, ..., \underbrace{\gamma e - y_{n_2} d_{\nu_0}}_{n_2},$$
$$\underbrace{-y_{n_2+1} d_{\nu_0}}_{n_2+1}, ..., \underbrace{\gamma e - y_{n_{n_0}} d_{\nu_0}}_{n_{n_0}}, \underbrace{-y_{n_{n_0}+1} d_{\nu_0}}_{n_{n_0}+1}, ..., \underbrace{(1 - y_{n_{n_0+1}}) d_{\nu_0}}_{n_{n_0+1}}, ...).$$

Now we are going to show that (in both cases (1) and (2))

$$\rho(P\mathbf{x}(b)) > \rho(\mathbf{x}(b)), \qquad \text{for every} \quad b \in [b_1, b_2]. \tag{2.11}$$

Consider the case (1), then $\gamma = -1$.

Take $l_1, ..., l_{n_0}$ and $j_1, ..., j_{2m}$ from Lemma 2.4, point (2). Modifying ν_0, if necessary, by (2.10) and Lemma 2.4 we have

$$
\begin{aligned}
\rho(P\mathbf{x}(b)) &\geqslant \rho((P\mathbf{x}(b))_{j_1,\dots,j_{2m}}) \\
&\geqslant f_{l_1}(|-y_{n_1+1}\,d_{v_0}+b+y_{n_1}\,d_{v_0}|) \\
&\quad + f_{l_2}(|-y_{n_2+1}\,d_{v_0}+e+y_{n_2}\,d_{v_0}|)+\cdots \\
&\quad + f_{l_{n_0-1}}(|-y_{n_{n_0-1}+1}\,d_{v_0}+e+y_{n_{n_0-1}}\,d_{v_0}|) \\
&\quad + f_{l_{n_0}}(|(1-y_{n_{n_0+1}})\,d_{v_0}+e+y_{n_{n_0}}\,d_{v_0}|) \\
&= f_{l_1}(b+\underbrace{(y_{n_1}-y_{n_1+1})\,d_{v_0}}_{=c})+f_{l_2}(e+\underbrace{(y_{n_2}-y_{n_2+1})\,d_{v_0}}_{\geqslant 0})+\cdots \\
&\quad + f_{l_{n_0-1}}(e+\underbrace{(y_{n_{n_0-1}}-y_{n_{n_0-1}+1})\,d_{v_0}}_{\geqslant 0}) \\
&\quad + f_{l_{n_0}}(e+d_{v_0}+\underbrace{(y_{n_{n_0}}-y_{n_{n_0+1}})\,d_{v_0}}_{\geqslant 0}) \\
&\geqslant f_{l_1}(b+cd_{v_0})+f_{l_2}(e)+\cdots+f_{l_{n_0-1}}(e)+f_{l_{n_0}}(e+d_{v_0}) \\
&> f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{n_0-1}}(e)+f_{l_{n_0}}(e+d_{v_0})+2\varphi_{n_0}(d_{v_0}) \\
&\geqslant \rho(\mathbf{x}(b)).
\end{aligned}
$$

Now consider the case (2), then $\gamma=1$.

Take $l_1, \dots, l_{n_0}$ and $j_1, \dots, j_{2m}$ from Lemma 2.4, point (1). Modifying v_0, if necessary, by (2.10) and Lemma 2.4 we have

$$
\begin{aligned}
\rho(P\mathbf{x}(b)) &\geqslant \rho((P\mathbf{x}(b))_{j_1,\dots,j_{2m}}) \\
&\geqslant f_{l_1}(|-y_{n_1+1}\,d_{v_0}-b+y_{n_1}\,d_{v_0}|) \\
&\quad + f_{l_2}(|-y_{n_2+1}\,d_{v_0}-e+y_{n_2}\,d_{v_0}|)+\cdots \\
&\quad + f_{l_{n_0}}(|-y_{n_{n_0}+1}\,d_{v_0}-e+y_{n_{n_0}}\,d_{v_0}|) \\
&= f_{l_1}(b+\underbrace{(y_{n_1+1}-y_{n_1})\,d_{v_0}}_{=c})+f_{l_2}(e+\underbrace{(y_{n_2+1}-y_{n_2})\,d_{v_0}}_{\geqslant 0})+\cdots \\
&\quad + f_{l_{n_0}}(e+\underbrace{(y_{n_{n_0}+1}-y_{n_{n_0}})\,d_{v_0}}_{\geqslant 0}) \\
&\geqslant f_{l_1}(b+cd_{v_0})+f_{l_2}(e)+\cdots+f_{l_{n_0}}(e) \\
&> f_{l_1}(b)+f_{l_2}(e)+\cdots+f_{l_{n_0}}(e)+2\varphi_{n_0}(d_{v_0})\geqslant \rho(\mathbf{x}(b)).
\end{aligned}
$$

Now, consider a function $t\colon b\mapsto \rho(\mathbf{x}(b))$. It can be easily seen that for a fixed d_{v_0} and e this function is continuous. And since, by Lemma 2.4, (2.6), and (2.9)

$$t(b_1)=\rho(\mathbf{x}(b_1))\leqslant \hat{f}_{n_1}(b_1)+(n_0-1)\,\psi(e)+\psi(e+d_{v_0})+2\psi(d_{v_0})$$
$$<2\hat{f}_{n_1}(b_1)<1,$$
$$t(b_2)=\rho(\mathbf{x}(b_2))\geqslant \hat{f}_{n_1}(b_2)>1,$$

therefore there is $b_0\in(b_1, b_2)$ such that

$$1=t(b_0)=\rho(\mathbf{x}(b_0)).$$

Thus for this b_0, by (2.11), we have

$$\rho(P\mathbf{x}(b_0))>\rho(\mathbf{x}(b_0))=1.$$

Hence, by Remark 1.12, $\|P\|>1$, and consequently P_0 is the only minimal projection and has norm equal to 1.

To prove the converse suppose $\varphi'_+(0)>0$ (by Corollary 1.4 $\varphi'_+(0)$ exists). Take $c\in(0, 1)$ from Corollary 1.10 for a function $h=f_1$. Put

$$\mathbf{y}_0=(1-c, 1, 1, ...), \tag{2.12}$$

and let

$$P\colon \mathcal{X}_{\mathcal{F}}\ni\mathbf{x}\mapsto\mathbf{x}-(\lim_{n\to\infty} x_n)\cdot\mathbf{y}_0\in\mathcal{Y}_{\mathcal{F}}. \tag{2.13}$$

By Remark 1.14, $P\in\mathcal{P}(\mathcal{X}_{\mathcal{F}}, \mathcal{Y}_{\mathcal{F}})$. Obviously $P\neq P_0$, since $\mathbf{y}_0\neq(1, 1, 1, ...)$ and there is $\mathbf{x}\in\mathcal{X}_{\mathcal{F}}\setminus\mathcal{Y}_{\mathcal{F}}$.

Take any $\mathbf{x}=\{x_n\}\in\mathcal{X}_{\mathcal{F}}$, $\rho(\mathbf{x})\leqslant 1$ and denote $d=\lim_{n\to\infty} x_n$. We show that

$$\rho(P\mathbf{x})\leqslant\rho(\mathbf{x}).$$

Without loss, we can assume that $d\neq 0$.

Fix any $\varepsilon>0$. By Lemma 1.11, we can take $1\leqslant j_1<j_2$, M_0 such that for any $M\geqslant M_0$ we can choose $K_0(M)$, $j_3, ..., j_{2M}$ such that

$$\rho_{\mathcal{F}}((P\mathbf{x})_{j_1, ..., j_{2M}, k})>\rho(P\mathbf{x})-\varepsilon \tag{2.14}$$

for every $k\geqslant K_0(M)$.

If $j_1\neq 1$, then by (2.14) we obtain

$$\rho(\mathbf{x})\geqslant\rho_{\mathcal{F}}(\mathbf{x}_{j_1, ..., j_{2M_0}, K_0(M_0)})=\rho_{\mathcal{F}}((P\mathbf{x})_{j_1, ..., j_{2M_0}, K_0(M_0)})>\rho(P\mathbf{x})-\varepsilon,$$

that is,

$$\rho(\mathbf{x})>\rho(P\mathbf{x})-\varepsilon.$$

Now assume that $j_1 = 1$. It will be shown that there exist $M_1 \geqslant M_0$, $K_1 \geqslant K_0(M_1)$ such that

$$f_1(|x_{j_2} - x_1|) + f_{M_1+1}(|x_{K_1}|) \geqslant f_1(|x_{j_2} - x_1 + cd|) + f_{M_1+1}(|x_{K_1} - d|), \tag{2.15}$$

where j_2, M_0, M_1, $K_0(M_1)$ are chosen from (2.14).

If not, then for any $M \geqslant M_0$, $K \geqslant K_0(M)$ $f_1(|x_{j_2} - x_1|) + f_{M+1}(|x_K|) < f_1(|x_{j_2} - x_1 + cd|) + f_{M+1}(|x_K - d|)$. Since $x_K \to d$, we get

$$f_1(|x_{j_2} - x_1|) + f_{M+1}(|d|) \leqslant f_1(|x_{j_2} - x_1 + cd|)$$

for any $M \geqslant M_0$.

But by the definition of φ there exists a sequence $\{M_l\}$ such that $f_{M_l}(|d|) \to \varphi(|d|)$, $l \to \infty$. Hence

$$f_1(|x_{j_2} - x_1|) + f_{M_l}(|d|) \leqslant f_1(|x_{j_2} - x_1 + cd|)$$

for any $l \in \mathbb{N}$. Passing with l to infinity, we get

$$f_1(|x_{j_2} - x_1|) + \varphi(|d|) \leqslant f_1(|x_{j_2} - x_1 + cd|). \tag{2.16}$$

By (2.16), $\varphi(|d|) < +\infty$. Since $f_1(|x_{j_2} - x_1|) < 2$ and $f_1(|d|) < 2$ (it follows from $\rho(\mathbf{x}) \leqslant 1$), by Corollary 1.10 we get

$$f_1(|x_{j_2} - x_1 + cd|) < f_1(|x_{j_2} - x_1|) + \varphi(|d|), \tag{2.17}$$

a contradiction with (2.16).

Now, for M_1 choose numbers $j_3, ..., j_{2m+1}$ from (2.14). Note that, by (2.13), (2.15) is equivalent to

$$\rho_{\mathscr{F}}(\mathbf{x}_{j_1, ..., j_{2M_1}, K_1}) \geqslant \rho_{\mathscr{F}}((P\mathbf{x})_{j_1, ..., j_{2M_1}, K_1}).$$

By (2.14),

$$\rho(\mathbf{x}) \geqslant \rho_{\mathscr{F}}(\mathbf{x}_{j_1, ..., j_{2M_1}, K_1}) \geqslant \rho_{\mathscr{F}}((P\mathbf{x})_{j_1, ..., j_{2M_1}, K_1}) > \rho(P\mathbf{x}) - \varepsilon.$$

Thus in both cases we have proved that $\rho(\mathbf{x}) > \rho(P\mathbf{x}) - \varepsilon$, for any $\varepsilon > 0$. By Remark 1.12, $\|P\| = 1$, and consequently P is a minimal projection different from P_0. ▮

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